1 Problem Statement and Introduction

Given the following system:

$$\dot{x} = 2x^2 + asin(x) + g(x) + u \tag{1}$$

where a is an unknown constant and $g(x)$ is an unknown function. Assume $a = 2$ when simulating the system itself (you dont use this value in the controller) and that $g(x) = 3cos(x)$ (you dont use this function in the controller but you will need this knowledge to develop a bounding function and simulate the system).

Design and simulate a single tracking controller that has three components: i) an exact model term, ii) an adaptive term, and iii) a robust term based on V_{R1}:

1. (10) Design the controller and show a stability analysis including your conclusion on expected tracking performance. Show all signals are bounded.
2. (10) Simulate the system in Simulink
   (a) Assume $x(0) = 5$ in the simulation
   (b) Document your simulation (show enough of your block diagram, etc)
   (c) The desired trajectory is given as $x_d = 2cos(3t)$.
      i. Plot the desired trajectory and tracking error on one plot. Plot 5 periods of the desired trajectory.
      ii. Plot the total control input and each of the three terms in your controller on one plot. Plot 5 periods of the desired trajectory.

2 Controller Design and Stability Analysis

First define the controller's tracking error:

$$e = x_d - x \tag{2}$$

Next, differentiate the error with respect to time and substitute in the definition of $\dot{x}$ from (1):

$$\dot{e} = \dot{x_d} - 2x^2 - asin(x) - g(x) - u \tag{3}$$

Next, it can be observed that a is unknown and linearly parameterizable; therefore, define $W = sin(x)$ and $\theta = a$ and substitute in $\dot{e}$ in order to set up our adaptive controller.

$$\dot{e} = \dot{x_d} - 2x^2 - W\theta - g(x) - u \tag{4}$$

Define the following as the Lyapunov function for this problem:

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} \tag{5}$$

Before taking the derivative of this, it is worth noting that $\tilde{\theta} = \theta - \hat{\theta}$.

Next, differentiate V with respect to time:

$$\dot{V} = e\dot{e} - \tilde{\theta}\dot{\hat{\theta}} \tag{6}$$

Substitute our definition of $\dot{e}$ in (4):

$$\dot{V} = e\left(\dot{x_d} - 2x^2 - W\theta - g(x) - u\right) - \tilde{\theta}\dot{\hat{\theta}} \tag{7}$$

A control can now be derived in order to stabilize our system.

$$u = \dot{x_d} - 2x^2 - W\hat{\theta} + V_{R1} + ke \tag{8}$$

Note, $2x^2$ is our exact model term. We do not know θ and therefore will use its estimate in this controller. Finally V_{R1} serves as the robust term. The stability analysis will be continued later; but first, the robust control term is discussed in more detail.

2.1 Robust Term Considerations

The V_{R1} term is the robust control term of this controller to deal with the unknown function $g(x)$. The robust control can be defined as:

$$V_{R1} = \rho\frac{e}{|e|} \tag{9}$$

Next, a bounding function for $g(x)$ is determined. For purposes of simulating this system we were asked to use:

$$g(x) = 3cos(x) \tag{10}$$

Therefore we can define our bounding function as:

$$|g(x)| \leq 3 = \rho \tag{11}$$

One must consider any uncertainty in the limits of $g(x)$ if actually implementing this controller on a physical system; however, for our simulations, this definition of ρ will capture the bounds of $g(x)$.

Finally, the control design can be written with the definition of our robust (sliding mode) term in (9).

$$\boxed{u = \dot{x_d} - 2x^2 - W\hat{\theta} + \rho\frac{e}{|e|} + ke} \tag{12}$$

Note that all three required terms are present, the exactly model knowledge ($-2x^2$), the adaptive term ($-W\hat{\theta}$), and the robust term ($\rho\frac{e}{|e|}$) plus the tracking control ($\dot{x_d}$) and a stabilizing term (ke).

2.2 Stability Analysis Continued

Now, back to the stability analysis. Next, our controller design from (12) can be substituted into $\dot{V}$ from (7).

$$\dot{V} = e\left(\cancel{\dot{x}_d} - \cancel{2x^2} - W\theta - g(x) - \cancel{\dot{x}_d} + \cancel{2x^2} + W\hat{\theta} - \rho\frac{e}{|e|} - ke\right) - \tilde{\theta}\dot{\hat{\theta}} \tag{13}$$

$$\dot{V} = e\left(-W\theta + W\hat{\theta} - g(x) - \rho\frac{e}{|e|} - ke\right) - \tilde{\theta}\dot{\hat{\theta}} \tag{14}$$

$$\dot{V} = -ke^2 - eW\left(\theta - \hat{\theta}\right) - e\left(g(x) + \rho\frac{e}{|e|}\right) - \tilde{\theta}\dot{\hat{\theta}} \tag{15}$$

$$\dot{V} = -ke^2 - \left(g(x)e + \rho\frac{e^2}{|e|}\right) - \tilde{\theta}\dot{\hat{\theta}} - \tilde{\theta}eW^T \tag{16}$$

$$\dot{V} = -ke^2 + \underbrace{\left(|g(x)||e| - \rho|e|\right)}_{\text{By definition (11)} \leq 0} - \tilde{\theta}\left(\dot{\hat{\theta}} + eW^T\right) \tag{17}$$

$$\dot{V} \leq -ke^2 - \tilde{\theta}\left(\dot{\hat{\theta}} + eW^T\right) \tag{18}$$

Finally, $\dot{\hat{\theta}}$ can be designed to cancel the remaining unwanted terms from $\dot{V}$

$$\dot{\hat{\theta}} = -eW^T \tag{19}$$

Or, from our definition of W:

$$\dot{\hat{\theta}} = -esin(x) \tag{20}$$

$\dot{V}$ can be reduced to the finally solution:

$$\dot{V} \leq -ke^2 \tag{21}$$

2.3 Stability Conclusion and System Bounds

Form our analysis V in (5) is Positive Definite (PD), radially unbounded and the final form of $\dot{V}$ in (21) is Negative Semi-Definite (NSD); therefore, the only conclusion that can be made at this point is the close loop system is stable, e and $\tilde{\theta}$ are bounded.

The closed loop error dynamics can be written as:

$$\dot{e} = \cancel{\dot{x}_d} - \cancel{2x^2} - W\theta - g(x) - \left(\cancel{\dot{x}_d} - \cancel{2x^2} - W\hat{\theta} + \rho\frac{e}{|e|} + ke\right) \tag{22}$$

$$\dot{e} = -W\tilde{\theta} - g(x) - \rho\frac{e}{|e|} - ke \tag{23}$$

Because $\tilde{\theta}$ and e are bounded and by definition $g(x)$ and ρ are also bounded, one can conclude that $\dot{e}$ is bounded.

Next, Barbalat's lemma can be applied to (21) to try and achieve a stronger sense of stability:

$$\ddot{V} = -2ke\dot{e} \tag{24}$$

Because e and $\dot{e}$ are bounded, $\ddot{V}$ is bounded therefore, $\dot{V} \to 0$ and in turn, $e \to 0$.

Next, because $\tilde{\theta}$ and e are bounded, the tracking function derivative is assumed to be bounded (and is in our simulation), and $g(x)$ and ρ are by definition bounded, u is bounded.

Because $e \to 0$, $x \to x_d$.

Because x, u, $g(x)$ are bounded, $\dot{x}$ is bounded.

2.3.1 Closed-loop system

Substituting our control in to our system definition and re-arranging terms shows how the different aspects of the control act on the system.

$$\dot{x} = 2x^2 + asin(x) + g(x) + \dot{x_d} - 2x^2 - W\hat{\theta} + \rho\frac{e}{|e|} + ke \tag{25}$$

$$\dot{x} = \underbrace{2x^2 - 2x^2}_{\text{Exact model knowledge}} + \underbrace{asin(x) - \hat{a}sin(x)}_{\text{Adaptive Term}} + g(x) + \underbrace{\rho\frac{e}{|e|}}_{\text{Robust Term}} + \dot{x_d} + ke \tag{26}$$

3 Simulation

3.1 Model

This problem was programed in the Simulink and the use of goto block were used in order to break the problem up into subsection for easier reporting. First, the plant was modeled from (1) and can be seen in Figure 1. Next, the desired trajectory is defined from the problem statement ($x_d = 2cos(3t)$ and $\dot{x_d} = -6sin(3t)$) can be seen in Figure 2. The parameter adaptation described in (19) can be seen in Figure 3. Finally, the system controller defined in (12) can be seen in Figure 4.

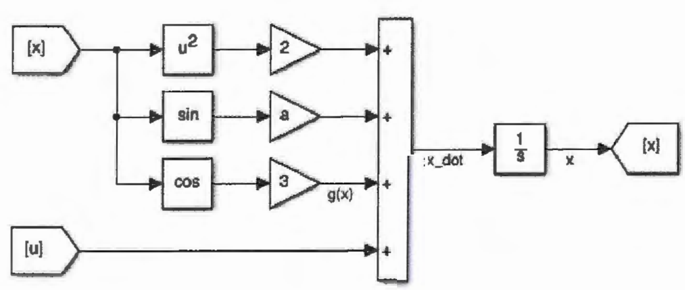

Figure 1: Simulink model of the plant.

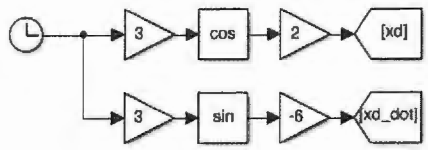

Figure 2: Tracking definition.

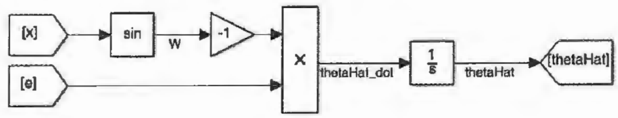

Figure 3: Simulink parameter adaptation.

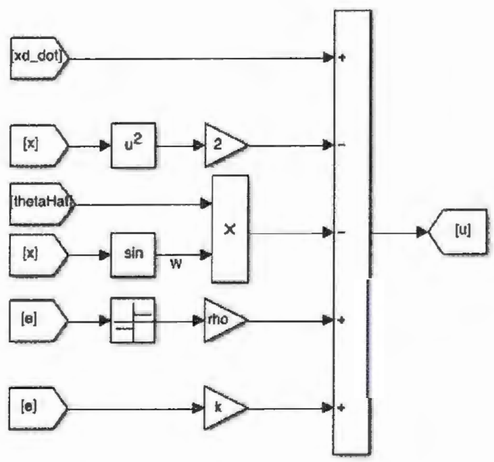

Figure 4: Simulink controller.

3.2 Results

The simulation was run using the following parameters:

- $a = 2$
- $k = 1$
- $\rho = 3$
- $x(t = 0) = 5$
- $\hat{\theta}(t = 0) = 1$

Figure 5 shows the system error tracking while Figure 6 describe the controller input its components. The tracking performance corroborates the stability analysis where the error from the desired trajectory is driven to 0.

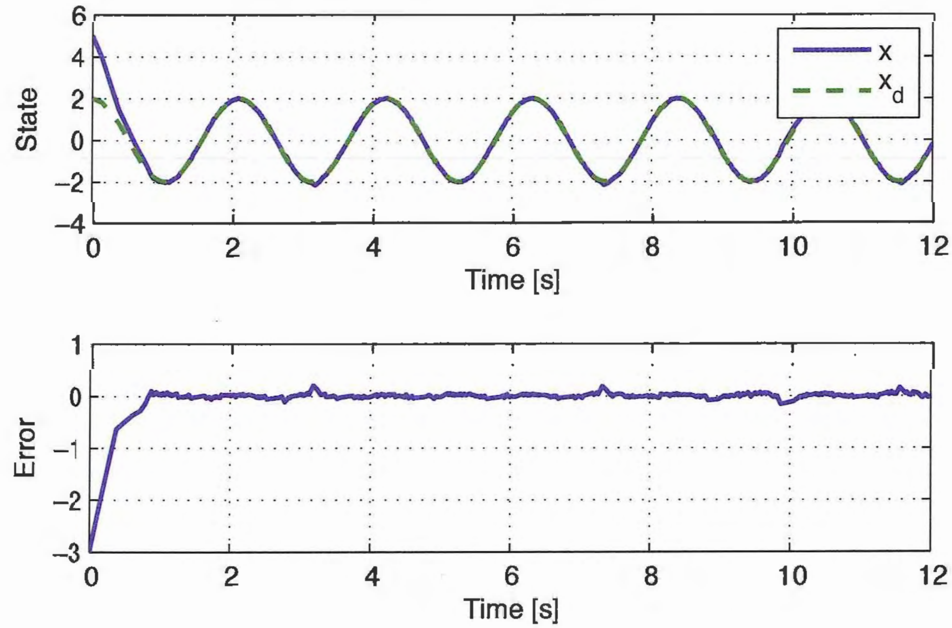

Figure 5: Error tracking.

4 Conclusion

In this problem, a control was design for the system described in (1) where the system had an exact knowledge portion, an unknown linear parameter, and an unknown disturbance with known bounds. A controller was designed to track an arbitrary input through a Lyapunov analysis using V (5) and $\dot{V}$ (21). An exact model knowledge term was used to cancel the terms that were known, an adaptive control strategy was constructed to provide an estimate of the unknown parameter and drive the tracking error to zero, and a robust control term was derived to cancel out

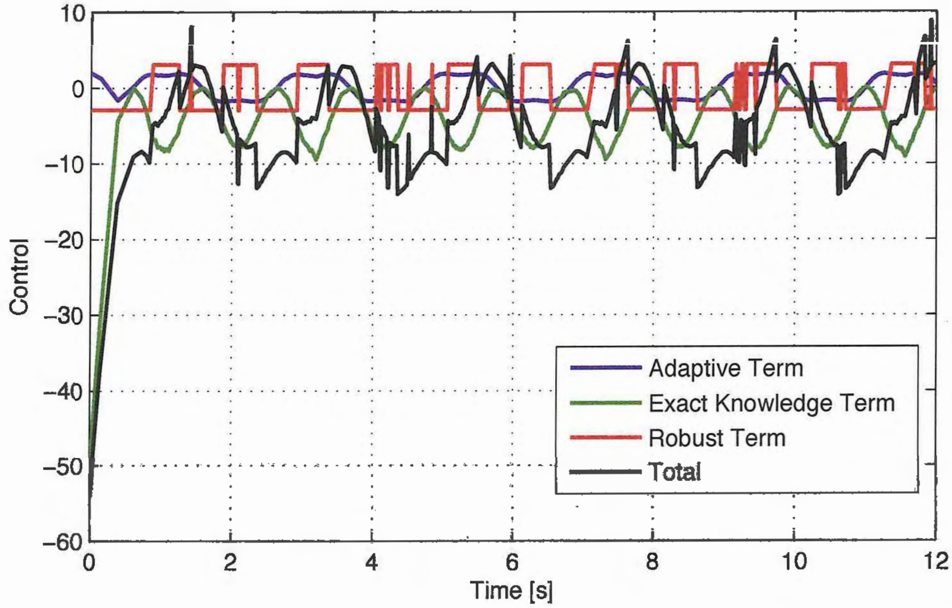

Figure 6: Controller input.

the unknown function $g(x)$. Through the stability analysis, it was proven that $e \rightarrow 0$ while all signals are bounded; in other words, the close loop system exhibited global asymptotic stability. Finally, this system was programed in Simulink to provide a demonstration of the control strategy and stability analysis.